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ORDERING OF THE ELEMENTS OF A MATROID SUCH THAT ITS CONSECUTIVE w ELEMENTS ARE INDEPENDENT

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Let M be a matroid on set E, $|E| = m$, with rank function r. For a positive integer w, M is said to be wth L-ind (C-ind) orderable if there exists an ordering O of E such that any consecutive (cyclically consecutive) w elements are independent. It is proved that M is wth L-ind orderable if and only if $\lfloor m/w \rfloor \cdot (w - r(E-S)) \leq |S| \leq \lceil m/w \rceil \cdot r(S)$ holds for any $S \subset E$. While, we conjecture that M is wth C-ind orderable if and only if $|S| \leq r(S) \cdot (m/w)$ holds for any $S \subset E$. This is verified for several classes.

1. Introduction

Let M be a matroid on set E with rank function r. We put $|E| = m$ and $r(E) = r$.

An ordering O of E is a bijection

$$O : E \to \{1, 2, \ldots, m\}.$$

For a positive integer w, O is said to be a wth L-ind (linearly independent) ordering if it satisfies the condition:

$$\text{(L)}\, A(i) = \{O^{-1}(k) \mid k = i, i+1, \ldots, i+w-1\}$$
$$\text{is independent for any } i, \quad 1 \leq i \leq m-w+1.$$

While if ordering O satisfies the following strengthened condition (C), O is said to be the wth C-ind (cyclically independent) ordering.

$$\text{(C)}\;\; B(i) = \{O^{-1}(k) \mid k = i, i+1, \ldots, i+w-1 \pmod m\}$$
$$\text{is independent for any } i, \quad 1 \leq i \leq m.$$

If there exists a wth L-ind (C-ind) ordering, M is said to be wth L-ind (C-ind) orderable. Note that if M is ith L-ind (C-ind) orderable, then for any $j \leq i$ M is jth L-ind (C-ind) orderable. Furthermore, the dual of an rth C-ind orderable matroid is $(m-r)$th C-ind orderable.

The purpose of this paper is to characterize these matroids. For the wth L-ind orderable matroids, a complete characterization is obtained. The proof is a direct extension of the previous work [1], which considers the problem in case $w = r$.

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However, for the wth C-ind orderable matroids, the problem looks very tough. We can only provide a necessary condition, called (w UNICOVER), though we conjecture that it is also sufficient. At present, we verify that the conjecture is true for the following cases.

(1) Graphs consisting of two disjoint spanning trees and any $w \leqslant r$.
(2) Matroids and $w = 1$ or 2.
(3) Simple graphs and $w = 3$ or 4.
(4) Complete graphs and any $w \leqslant r$.
(5) 2-trees and any $w \leqslant r$.

It must be noted that any matroid in (1), (4), and (5) satisfies (w UNICOVER) for any $w \leqslant r$.

2. wth L-ind orderable matroids

An independent set of cardinality w ($w \leqslant r$) is called for simplicity a w-set. Let $p = \lfloor m/w \rfloor$, $t = \lceil m/w \rceil$. The following two lemmas should be called the packing and covering theorems [2, 3] applied to the truncation of M to w, respectively.

Lemma 1. *There exist p disjoint w-sets in M if and only if the following condition is satisfied*:

$$(w\,PACK) \quad \text{For any } S \subset E,\ |S| \geqslant p \cdot (w - r(E - S)).$$

Lemma 2. *E is covered with t w-sets if and only if the following condition is satisfied*:

$$(w\ COVER) \quad \text{For any } S \subset E,\ |S| \leqslant t \cdot r(S).$$

The following lemma can also be proved by an analogous manner as for the above lemmas.

Lemma 3. *If M satisfies both (w PACK) and (w COVER), M is partitioned into p disjoint w-sets and one (possibly empty) independent set.*

Now we can give a characterization theorem which provides a polynomial time algorithm to recognize the wth L-ind orderable matroids.

Theorem 1. *The following four conditions for M are equivalent*:

(1) *Both (w PACK) and (w COVER) are satisfied, that is, for any $S \subset E$,*

$$p \cdot (w - r(E - S)) \leqslant |S| \leqslant t \cdot r(S)$$

(2) *E is covered with t w-sets and contains p disjoint w-sets.*
(3) *E is partitioned into p disjoint w-sets and one independent set.*
(4) *M is wth L-ind orderable.*

Proof. It suffices to prove (3)→(4). Let $E = A \cup S_1 \cup \cdots \cup S_p$ be the partition as in (3) where S_i is a w-set and A, $0 \leq |A| < w$, is an independent set.

We first show a method of ordering $A \cup S_1$ such that its consecutive w elements are independent. Then the idea is applied to extend the ordering of $A \cup S_1 \cup S_2$, $A \cup S_1 \cup S_2 \cup S_3, \ldots$, and $A \cup S_1 \cup \cdots \cup S_p$.

An element which is assigned the kth order is denoted by e_k. First of all, we give A with arbitrary ordering. Let $|A| = q$ $(< w)$ and $A = \{e_1, \ldots, e_q\}$. $S_1 \cup \{e_q\}$ has at most one circuit and the circuit has non-null intersection with S_1, since A and S_1 are both independent. Choose any element from S_1 which is contained in the circuit if one exists, and any element otherwise. And let it be e_{q+w}. Then, $S_1^1 = S_1 \cup \{e_q\} - \{e_{q+w}\}$ is a w-set. $S_1^1 \cup \{e_{q-1}\}$ has at most one circuit which has non-null intersection with S_1-$\{e_{q+w}\}$ since S_1 and A-$\{e_{q+w}\}$ are both independent. Choose any element from S_1^1 which is contained in the circuit if one exists, and any element otherwise. Let the element be e_{q+w-1}. Then, $S_1^2 = S_1^1 \cup \{e_{q-1}\} - \{e_{q+w-1}\}$ is a w-set.

Continue this procedure up to get the ordering

$$\{e_{1+w}, \ldots, e_{q+w}\} \subset S_1.$$

The rest, S_1-$\{e_{1+w}, \ldots, e_{q+w}\}$, is ordered arbitrarily from $q+1$ through w. Thus we obtain an ordering of $A \cup S_1$. It is evident that every set of consecutive w element is a w-set.

The above ordering was based only on the facts that A and S_1 are both independent and that $|S_1| = w$. Furthermore, recall that in the procedure we are allowed to order A arbitrarily. Thus we see that the same principle applies to the ordering of $S_i \cup S_{i+1}$ when S_i is ordered and S_{i+1} is not $(i = 1, \ldots, p-1)$. Thus, the ordering is extended to $A \cup S_1 \cup \cdots \cup S_p$. □

3. *w*th C-ind orderable matroids

If M is wth C-ind orderable, there exist m w-sets which cover E such that each element is covered exactly w times.

Consider a matroid $M^{(w)} = (E^{(w)}, r^{(w)})$ which is derived from M by replacing each element with w parallel elements. Note that $r^{(w)}(S^{(w)})$, $S^{(w)} \subset E^{(w)}$, is equal to $r(S)$ where $S \subset E$ consists of the elements whose corresponding elements in $M^{(w)}$ are in $S^{(w)}$. Then, if M is wth C-ind orderable, $M^{(w)}$ is covered with m w-sets and also contains m disjoint w-sets. Therefore, by Lemma 1 and 2,

$$\lfloor |E^{(w)}|/w \rfloor \cdot (r^{(w)}(E^{(w)}) - r^{(w)}(E^{(w)} - S^{(w)})) \leq |S^{(w)}| \leq \lceil |E^{(w)}|/w \rceil \cdot (r^{(w)}(S^{(w)})).$$

holds for any $S^{(w)} \subset E^{(w)}$. Since $|E^{(w)}| = wm$, this leads to

$$(m/w) \cdot (r - r(E - S)) \leq |S| \leq (m/w) \cdot r(S) \quad \text{for any } S \subset E.$$

In contrast to the linear case, the above two inequalities are equivalent [4]. Thus we get a necessary condition of the wth C-ind orderable matroids which is able to be checked in polynomial time [2]. We believe that this condition is also sufficient. Indeed, we have various affirmative examples for the conjecture, which we describe in primal form in the following.

Conjecture. A matroid M is wth C-ind orderable if and only if

$$(w\ \text{UNICOVER}) \quad |S| \leqslant (m/w) \cdot r(S) \quad \text{for any } S \subset E$$

is satisfied.

Note that M satisfies (r UNICOVER) if and only if its dual satisfies (r UNICOVER).

In the following, we list several matroids satisfying the condition for which we have been able to give wth C-ind orderings.

4. Examples of wth C-ind orderable matroids

For a graph $G=(V, E)$, $G-S$, $S \subset V$, is the graph obtained from G by deleting S. $E(G)$ also denotes the set of edges of G. An edge connecting vertices v and u is denoted by (v, u).

A graph consisting of two disjoint spanning trees is called a CTS-graph (complementary tree structure graph) and one of the trees is called a peripheral tree. It is easy to see that any CTS-graph satisfies (w UNICOVER) for any $w \leqslant r$.

Theorem 2 [1]. *A CTS-graph is wth C-ind orderable for any $w \leqslant r$.*

Proof. It suffices to prove the theorem for $w=r$. We shall apply induction on r. If $r=1$ then the theorem is obviously true. (See Theorem 3.) Suppose that $r \geqslant 2$. Note that G contains a vertex of degree 2 or 3. If there exists a vertex of degree 2, say v, G-$\{v\}$ is again a CTS-graph. By induction hypothesis, there is an $(r-1)$th C-ind ordering O' in G-$\{v\}$. Define an ordering O of edges of G as follows.

$$O(e)=\begin{cases} O'(e) & (e \in E(G-\{v\}) \text{ and } O'(e) \leqslant r-1) \\ O'(e)+1 & (e \in E(G-\{v\}) \text{ and } O'(e) \geqslant r) \\ r & (e \text{ is one edge incident to } v) \\ 2r & (e \text{ is the other edge incident to } v). \end{cases}$$

It is easy to see that any cyclically consecutive r edges under O are independent.

Suppose that there is a vertex of degree 3, say v. Let x, y, and z be the vertices adjacent to v. Two of them may be identical. Assuming that (x, v) and (y, v) are contained in a peripheral tree, let G' be the graph obtained from G by removing vertex v and adding an edge (x, y). G' is also a CTS-graph. Hence there is an

$(r-1)$th C-ind ordering O'. Without loss of generality, we can assume $O'((x, y)) = 1$. Let $P = \{O'^{-1}(2), O'^{-1}(3), \ldots, O'^{-1}(r-1)\}$. Since $P \cup \{(x, y)\}$ is a spanning tree of G', exactly one of x and y, say x, is connected to z in P. Now define an ordering O of edges of G as follows.

$$O(e) = \begin{cases} O'(e) & (e \in E(G') \text{ and } 2 \leqslant O'(e) \leqslant r-1) \\ O'(e)+1 & (e \in E(G') \text{ and } O'(e) \geqslant r) \\ 1 & (e = (v, a)) \\ r & (e = (v, c)) \\ 2r & (e = (v, b)). \end{cases}$$

It is easy to see that O is an rth C-ind ordering. □

Since it is trivial that matroid M satisfies (1 UNICOVER) if and only if M contains no loop, the following fact is evident.

Theorem 3. *A matroid M is* 1*st C-ind orderable if and only if M satisfies* (1 *UNICOVER*).

Since it is easy to see that matroid M satisfies (2 UNICOVER) if and only if $r \geqslant 2$ and any $S \subset E$ such that $r(S) = 1$ contains at most $\frac{1}{2}m$ elements, it is not difficult to prove the following theorem.

Theorem 4. *A matroid M is* 2*nd C-ind orderable if and only if M satisfies* (2 *UNICOVER*).

Lemma 4. *A simple graph G satisfies* (3-*UNICOVER*) *if and only if $r \geqslant 3$ and G is not the graph with $m = 4$ containing a triangle.*

Proof. The necessity is obvious. Suppose that G does not satisfy (3-UNICOVER). If $r \geqslant 3$, there exists S such that

$$|S| > (m/3) \cdot r(S).$$

Since $m \geqslant |S|$, $2 \geqslant r(S) \geqslant 1$. If $r(S)$ is 1 or 2, then $|S|$ is at most 1 or 3, respectively. By the inequality, $m \leqslant 2$ or $m \leqslant 4$, respectively. From $r \geqslant 3$ and $m \geqslant r + |S| - r(S)$, only the case when the above inequality holds is $m = 4$, $r(S) = 2$, and $|S| = 3$. □

Theorem 5. *A simple graph G is* 3*rd C-ind orderable if and only if G satisfies* (3 *UNICOVER*).

Proof. The theorem is obviously true when $r = 3$. Furthermore, it is not difficult to check that every simple graph is 3rd C-ind orderable if $r = 4$. With using this fact as basis, we prove the theorem by induction on r. Suppose that $r(G) \geqslant 5$, $|V| = n \geqslant 6$. Let δ be the minimum degree of G. Then, $\delta \leqslant \frac{1}{3}m$ since $\delta n \leqslant 2m$. Let v be a vertex of degree at most $\frac{1}{3}m$. By the induction hypothesis, there exists a 3rd C-ind ordering in G-$\{v\}$. We can insert the edges which are incident to v into this ordering so that no pair of these edges are within distance 3 to get a 3rd C-ind ordering of G. □

The following lemma seems not so trivial but the proof is omitted here for the space.

Lemma 5. *A simple graph G satisfies* (4 *UNICOVER*) *if and only if $r \geqslant 4$ and G does not contain K_3 if $m = 5$, $K_{1,1,2}$ if $m = 6$, or K_4 if $m = 7$ as an induced subgraph.*

Theorem 6. *A simple graph $G = (V, E)$ is* 4*th C-ind orderable if and only if G satisfies* (4 *UNICOVER*).

Proof. It is so cumbersome to describe our proof in detail that we will sketch it. First, we prove that any connected simple graph with 7 vertices is 4th C-ind orderable, almost through the exhaustive way. Then, we can prove the fact that any connected simple graph which has more than 7 vertices is 4th C-ind orderable, completing the proof. This key fact is proved by induction on n as follows. Suppose that G has n (> 7) vertices. From $\delta n \leqslant 2m$ and $n \geqslant 8$, it is concluded that G contain a vertex of degree at most $\frac{1}{4}m$, say v. By induction hypothesis, the edges of G-$\{v\}$ can be so ordered that any cyclically consecutive 4 edges are independent. We can insert the edges which are incident to v into this ordering to get an ordering of E so that no pair of these edges are within distance 4. □

It is easy to see that a complete graph satisfies (r UNICOVER). Illustrative examples in Fig. 1 and 2 showing a way of rth C-ind ordering will be the proof of the following theorem.

Theorem 7. *A complete graph is wth C-ind orderable for any $w \leqslant r$.*

For a positive integer k, a k-tree is the graph defined recursively as follows: (1) K_k is a k-tree, (2) a graph obtained from a k-tree T_k by adding a vertex v and edges that connect v and the vertices forming a K_k in T_k is a k-tree.

Lemma 6. *A k-tree satisfies* (*r-UNICOVER*).

The proof is omitted here for reasons of space. But for the case of 2-trees, the following theorem implies the lemma.

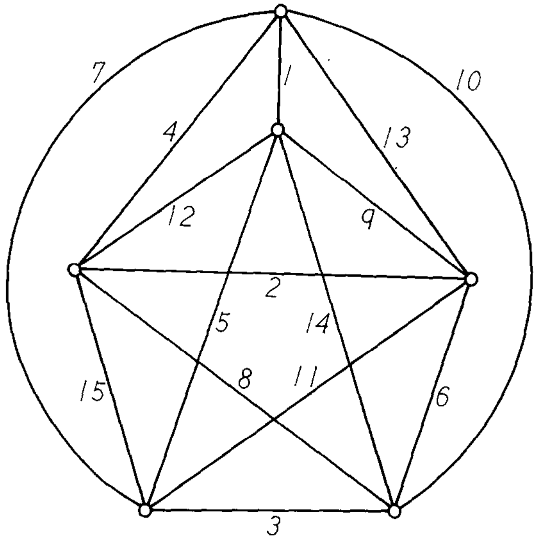

Fig. 1. An rth C-ind ordering of K_6.

Theorem 8. *A 2-tree is wth C-ind orderable for any $w \leqslant r$.*

Proof. We shall apply induction on r. If $r = 1$, that is K_2, then our theorem is true. Suppose that $r \geqslant 2$. By the definition of 2-tree, we can find a vertex v which was added at the last stage of construction. Let u and w be the vertices which are adjacent to v. G-$\{v\}$ is a 2-tree of rank $r - 1$, and so is $(r-1)$th C-ind orderable.

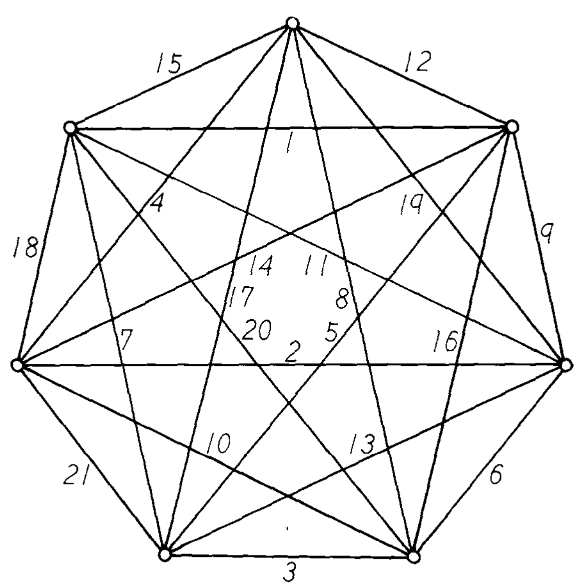

Fig. 2. An rth C-ind ordering of K_7.

Let O' be such an ordering and $O'((u, w)) = 1$. Define an ordering of edges of G as follows:

$$O(e) = \begin{cases} 1 & (e = (u, w)) \\ 2 & (e = (u, v)) \\ O'(e) + 1 & (e \in E(G - \{v\}) \text{ and } 2 \leqslant O'(e) \leqslant r - 1) \\ r + 1 & (e = (w, v)) \\ O'(e) + 2 & (e \in E(G - \{v\}) \text{ and } r \leqslant O'(e) \leqslant 2r - 3). \end{cases}$$

Every set of cyclically consecutive r edges under O that contains just one of (u, v) and (w, v) is independent. The only exception is set $S = \{(u, v\}, O^{-1}(3), \ldots, O^{-1}(r), (w, v)\}$. This is also independent because $S \cup \{(u, w)\} - \{(u, v), (w, v)\}$ is an $(r-1)$-set of G-$\{v\}$. $\square$

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